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Original Research Article

Optimal Portfolio Allocation with Price Limit Constraint

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Price limits set up are adopted by many securities markets in countries such as the USA, Canada, Japan, and various other countries in Europe and Asia, to increase the stability of the financial market. These limits confine the price of the financial asset during any trading day to a range, usually determined based on the previous day's closing price. In this paper, we study the portfolio optimization problem while taking into account the price limit constraint. The dynamic programming technique is applied to derive the Hamilton–Jacobi–Bellman equation, and the method of Lagrange multiplier is used to tackle the constraint. Optimization problem solution results and numerical method show that the equilibrium path of wealth and investment in risky assets has a different pattern than the absence of price limits.

Keywords: Optimal Portfolio; Limited Prices; Dynamic Programming.
JEL Classification*:* C58, G11

1 Introduction

Imposing the price limit constraint in the stock markets of developed countries such as the U.S., Japan, Canada, and some of the major economies of Europe and Asia Trading has been accepted as a circuit breaker.

One of the primary goals in establishing the stock market in emerging economies is to create a transparent mechanism to determine the price of financial assets. Besides, excited behavior or herd behavior is more prevalent among investors in emerging markets. One of the most important rules in this area imposed to control these behaviors is the price limit as a tool against large fluctuations in stock prices.

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Also, one of the most critical questions in investment management is the suitable amount of wealth invested in any assets by considering the circuit breaks.

The basic approach for investor's portfolio optimization in these areas is mainly known as the mean-variance approach presented by Markowitz (1952) as a pioneer in portfolio optimization. The study includes a single-period model, in which investors decide, at the beginning of the period, how to invest in different assets and hold the portfolio by the end of the period.

The investigation eventually led to the development of one period model to the continuous-time models. In these models using stochastic control theory for the optimization problem in special cases results in obtaining explicit solutions.

Imposing a limited range of prices by considering the optimization of the asset allocation problem is causing changes in the equilibrium pathway. This difference is due to the difference between the stock market price and the equilibrium price. Considering the mentioned circumstances in this paper, we seek an answer to the question of whether the optimization of asset allocation in an investor's portfolio with dynamic constraint (limited to price fluctuations) is different from an unconstrained state or not?

Generally, price limit impacts the overall trend of the market and increase market liquidity risk. Most research has focused on utility maximization and the final wealth, which will review with more details in the next part.

The objective function of investors may be different for many of them, and some of the investors in the stock market don't use the objective function like expected utility maximization in consumption for their decision. In other words, many investors tend to grow and maximize the value of their wealth with the degree of risk aversion than to buy bonds or deposit their money in the bank.

In this case, the objective function of these investors is defined as follows: investors expected utility is based on utility theory, which was proposed by Bernoulli (1738, 1954) and Von Neumann and Morgenstern (1944), which includes the amount of wealth with the capture of risk aversion. Thus, the optimization problem of an investment company (investor) with the objective function of maximizing the wealth is applied when the limited price is imposed for risky assets.

The dynamic expected motion of wealth is followed by an investor. Furthermore, after substitute the optimization problem with Bellman Jacob Hamilton (HJB) equation, then the optimization problem with constraint will be solved using the Lagrangian approach. Finally, using described methods, optimal solutions, and the optimal paths by limited price and without limitation will be compared. Eventually, using numerical methods for solving the HJB equation with constraint, the optimal strategy for these problems are compared together.

The area of portfolio optimization has changed a lot in analysis and measurements. For example, in recent years, this study is about how the investor can use some new methods and technology like machine-learning and reverse quantum to asset allocation and optimization. (Kolm et al., 2014; Fouquet et al., 2017, Venturelli and Kondratiev, 2019).

The rest of the article is organized as follows. In the second part, the empirical literature is studied to investigate the effect of applying different constraints and price limits on investors' portfolios or an investment company.

In the third section, using different methods, companies' optimal motion in the balance sheet for allocation of wealth in risky assets is optimized, and numerical simulation for the simple problem will be introduced. In the fourth part, the effect of this policy on the stock exchange market is concluded.

2 Literature Review

As mentioned in the introduction, initially, portfolio optimization techniques were assessed as a one-period problem. Merton (1969 and 1971) extended this method to the continuous-time model in his investigations. Using stochastic control theory for the portfolio optimization problem led to an implicit answer in some particular cases for the proportion of each asset in the portfolio. Due to the increasing financial depth in capital markets, development of the financial instruments, derivatives, and trading fees on some financial products, investors' exposure to the risks can be several times more than the initial investment allocating to that portfolio. A part of the investment may be financed as the initial capital investor (shareholder), and the other part is financed by different methods. Many types of research on portfolio optimization used constraints such as liquidity constraints and limited value at risk. Among These researches, Cvitanic and Karatzas (1998), Karatzas and start (1998), and Cui et al. (2011) and Yu (2004), and Yu et al. (2012) can be pointed out. Cvitanic and Karatzas (1998) used a Martingale process to optimize investment portfolios with the same variance for the case of two risky assets when the borrowing constraints are present.

Grossman and Vila (1992) used a stochastic dynamic model to analyze the value of Merton's optimal portfolio for the standard model in the presence of liquidity constraints. The model used in the study of Merton maximizes the expected utility of consumption for a specified period, which resulted in the allocation of wealth among asset risk, non-risk, and consumption. In Merton's study, the value at risk of assets did not consider as a limitation, so that it may be binding in that period. Primarily if a power utility function for maximization of the investor's expected utility is used, according to the optimal allocation in this article, which is a constant proportion of investor wealth, in some cases, it violates the value at risk constraint.

In the literature, different researchers studied the optimization technique of mean -Value at risk (mean-VaR) approach and compared them with the mean-variance (mean-variance) method. Kluppelberg and Korn (1998), Alexander and Baptista (1999), Kast, Luciano, and Peccati (1999), Dybvig et al. (2010), and Black et al. (2002) works are some of these researchers. Note that the models used in these studies are mostly static. Some Papers have been done by dynamic modeling. Examples are the works of Luciano (1998), Basak and Shapiro (2001), Yu (2004), Pliska (1997), Kamein, and Schwartz (1991), and Yu et al. (2010).

Yu et al. (2010), considering a dynamic programming model that constrains with the limit value at risk and numerical solution, show that when the return of risky asset comes from a Markovian Brownian motion, asset allocation are very sensitive to value at risk parameters and switching between the regimes. Researchers' idea in this article is the Maximum Value-at-Risk (MVaR) amount in a short time interval with different regimes.

Kolm et al. (2014) study the approaches for implementing Markowitz's mean-variance analysis. Also, their study covers the inclusion of transaction costs, constraints, and sensitivity to inputs and point out the new trends and developments in the area such as diversification methods, risk-parity portfolios, the mixing of different sources of alpha, and practical multi-period portfolio optimization.

Ouquet et al. (2017) study the Merton portfolio optimization problem in the presence of stochastic volatility using asymptotic approximations when the volatility process is characterized by its timescales of fluctuation. This approach is tractable because it treats the incomplete market problem as a perturbation around the complete market constant volatility problem for the value function, which is well understood.

Venturelli and Kondratiev (2019) investigate a hybrid quantum-classical solution method to the mean-variance portfolio optimization problems. They start from real financial data and follow the principles of the Modern Portfolio Theory. They generate parametrized samples of portfolio optimization problems related to quadratic binary optimization forms programmable in the analog D-Wave Quantum Annealer 2000Q™.

3 Optimization of Asset Allocation

In this study, the T is terminal time and $(\Omega, \Im, P, \{\Im_t\}_{t\geq 0})$ is the set of probability filters, considering the amount of wealth $W(t) = (W_1(t), \dots, W_n(t))'$ as a Brownian process $W(0) = 0, \Im_t = \sigma\{W(s); s \leq t\}$.

Besides, the process $X(.) = \{X(t); 0 \leq t \leq T\}$ is a stochastic process adopted to $\Im$and assume that $E \int_0^T |X(t)|^2 \, dt < +\infty$. Considering a dynamic model for a financial market that is defined on the time interval $[0, T]$, we assume that total wealth (asset) of an investment company (investor) at time t is given by $W(t) = X(t)$. The followings are the amount of wealth in a risky asset (stock market) $S(t) = \pi_1 X(t)$ and risk-free assets such as bonds $B(t) = \pi_2 X(t)$ and cash amount $(t) = \pi_3 X(t)$, which can be ignored.

Then suppose$\pi_1 + \pi_2 = 1$ according to the value of the total assets of the investment company. In the case of a continuous-time process, the process of these assets in the investment problem is written in the form of ordinary differential equations and stochastic differential equations.

It means $d\big(B(t)\big) = rB(t)\text{dt}$ is the moving process for bond value and $d\big(S(t)\big) = S(t)\{\mu \, \text{dt}+\sigma \, \text{dW}\}$ is the moving process for risky assets value at $t \in [0, T]$. In the above Differential equations, *r* is the risk-free rate of return on assets, μ is the return on average assets, and σ is the variance of the risky asset.

Using the relationship between the risky assets equation and wealth, we can rewrite the wealth process motion as follow:

$$d\big(X(t)\big) = \left\{\mu \underbrace{\pi_1 X(t)}_{\pi_1^*} + r\pi_2 X(t)\right\} \text{dt}+\pi_1 X(t)\sigma \, \text{dW}$$
$$=rX(t) + (\mu - r)\pi_1^* + \pi_1^*\sigma \, \text{dW}$$

3.1 Investor's Wealth Path under Limited Price Constraint

Von Neumann's utility function for investor's wealth is a function that represents the risk orientation of investors. We assume that the expected utility function of the investor's wealth is a power utility function of wealth. According to Arrow (1971) and Pratt's (1964) researches, relative risk aversion and absolute risk aversion coefficient for wealth utility function $U(W)$ are shown, respectively, by $RRA(W) = \frac{-WU''(W)}{U'(W)}$ and $ARA(W) = \frac{-U''(W)}{U'(W)}$.

Given the above, the objective function to maximize investor wealth under the dynamic process of wealth and price limit constraint for a risky asset is the optimization problem given in Equation (1).

$$\underset{\pi_1^*}{Max}\ \mathrm{E}\int_0^T e^{-rt}\frac{(X(t))^{1-\gamma}}{1-\lambda}\,dt$$
$$s.t\ \begin{cases} d(X(t)) = rX(t) + (\mu - r)\pi_1^* + \pi_1^*\sigma\,\mathrm{dW} \\ -k \le \pi_1 X(t)\{\mu\,\mathrm{dt}+\sigma\,\mathrm{dW}\} \le k \end{cases} \tag{1}$$

In this equation, the constant k> 0 in the price limit imposes price ceiling and floor in the stock market and $e^{-rt}\frac{(X(t))^{1-\gamma}}{1-\lambda}\,dt$is the discounted amount of expected wealth with a free risk bond return that has been given.

To obtain optimal solutions for Equation (1), the HJB method is used. It is assumed that $\pi_1 X(t)$ is the value of the risky asset at time t and $X(t)$ is the total amount of wealth. Using a dynamic programming approach for the problem (1) and find the solution is equivalent to find the solution by the HJB method. (Campbell et al., 2001; Bjork, 1998 and Oksendal, 2002)

We define the value function $J(x,t) = \underset{\pi_1^*}{Max}\ \mathrm{E}\int_0^T e^{-rt}\frac{(X(t))^{1-\gamma}}{1-\lambda}\,dt$. Using Ito lemma, we can drive Equation (2) as follow:

$$dJ(x,t) = [J_t + J_x(rX(t) + (\mu - r)\pi_1^*) + \frac{1}{2}J_{xx}(\pi_1^*)^2\sigma^2]\mathrm{dt} + J_x\pi_1^*\sigma\mathrm{dW} \tag{2}$$

Then we get the HJB Equation (3) as follow:

$$\begin{cases} J_t + J_x(\mu - r)\pi_1^* + \frac{1}{2}J_{xx}(\pi_1^*)^2\sigma^2 + rxJ_x + e^{-rt}\frac{x^{1-\gamma}}{1-\gamma} = 0 \\ J(x,T) = 0 \end{cases} \tag{3}$$

To solve the HJB equation, the static optimization problem changes to Equation (4):

$$\underset{\pi_1^*}{min}\, J_x(\mu - r)\pi_1^* + \frac{1}{2}J_{xx}(\pi_1^*)^2\sigma^2$$
$$s.t\ -k \le \pi_1^*\{\mu\,\mathrm{dt}+\sigma\,\mathrm{dW}\} \le k \tag{4}$$

To solve the problem (4), we should note that the restriction is the stochastic process or stochastic differential equation for the risky asset.

In other words, due to the stochastic nature of this constraint, it cannot be directly solved in a Lagrange equation. To solve the optimization problem, we consider using the probability distribution function of the constraint stochastic process for the different levels of significance. Due to the increment of the Brownian process, which is a function of the normal distribution, the mean and variance distribution function of a risky asset are $E(S_t - S_0) = \pi_1^* \mu t$ and $Var(S_t - S_0) = (\pi_1^* \sigma t)^2$ respectively.

The dynamic continuous-time equation for the risky asset as a probability function is $p(-k < \pi_1^*[\mu dt + \sigma dz] < k) = \gamma$ and can be rewritten in the form of $\left(\frac{-\frac{k}{\pi_1^*} - \mu dt}{\sigma} < dz < \frac{\frac{k}{\pi_1^*} - \mu dt}{\sigma} \right)$.

dz is increment amounts of a risky asset, which with the standardization process, can be rewritten as the difference of two normal cumulative distribution functions. To solve the above optimization problem, we can write down the Lagrange function the same as Equation (5):

$$L(\pi_1^*, \lambda_1(x,t), \lambda_2(x,t)) = J_x(\mu - r)\pi_1^* + \frac{1}{2} J_{xx}(\pi_1^*)^2 \sigma^2$$
$$+\lambda_1(x,t)\left\{\frac{k}{\sigma \Phi^{-1}(\gamma_1) + \mu dt} - \pi_1^*\right\} + \lambda_2(x,t)\left\{\frac{k}{\sigma \Phi^{-1}(\gamma_2) + \mu dt} + \pi_1^*\right\} \quad (5)$$

After obtaining the first-order conditions, the optimal values for π_1^*, λ_1^* and λ_2^* states are the same as equation (6):

$$\frac{\partial L}{\partial \pi_1^*} = 0 \Rightarrow \pi_1^* = \frac{J_x(r-\mu)-\lambda_1+\lambda_2}{J_{xx}\sigma^2}$$
$$\lambda_1(x,t)\left\{\frac{k}{\sigma\Phi^{-1}(\gamma_1)+\mu dt} - \pi_1^*\right\} = 0$$
$$\lambda_2(x,t)\left\{\frac{k}{\sigma\Phi^{-1}(\gamma_2)+\mu dt} + \pi_1^*\right\} = 0 \tag{6}$$
$$\begin{cases} \lambda_1 < 0, \lambda_2 = 0 \Rightarrow \pi_1^* = \frac{k}{\sigma\Phi^{-1}(\gamma_1)+\mu dt} \\ \lambda_2 < 0, \lambda_1 = 0 \Rightarrow \pi_1^* = -\frac{k}{\sigma\Phi^{-1}(\gamma_2)+\mu dt} \\ \lambda_2 < 0, \lambda_1 < 0 \Rightarrow \times \\ \lambda_2 = 0, \lambda_1 = 0 \Rightarrow \pi_1^* = \frac{J_x(r-\mu)}{J_{xx}\sigma^2} \end{cases}$$
$$\lambda_1^* = \left(\frac{kJ_{xx}\sigma^2}{\sigma\Phi^{-1}(\gamma_1)+\mu dt} + J_x(r-\mu)\right)$$
$$\lambda_2^* = \left(\frac{-kJ_{xx}\sigma^2}{\sigma\Phi^{-1}(\gamma_1)+\mu dt} + J_x(r-\mu)\right)$$

With substituting the optimal value π_1^* in Equation (4), optimal value function $J^*(x,t)$ can be calculated. Because the answer to the Equations (3) and (6) are nonlinear for $J^*(x,t)$ and π_1^*, we use the numerical method to find the optimal solution.

If there is no constrain such as limited prices, the optimization problem with the HJB method does not include these constraints and $\pi_1^* = \frac{J_x(r-\mu)}{J_{xx}\sigma^2}$ is the optimal solution. The HJB equation can be rewritten as Equation (7).

$$\begin{cases} J_t - \frac{1}{2}\frac{J_x^2(r-\mu)^2}{J_{xx}\sigma^2} + rxJ_x + e^{-rt}\frac{x^{1-\gamma}}{1-\gamma} = 0 \\ J(x,T) = 0 \end{cases} \tag{7}$$

To solve the above ordinary differential equations, we assume that the functional form $J(x,t)$ is the same as Equation (8):

$$J(x,t) = f(t)\frac{x^{1-\gamma}}{1-\gamma} \tag{8}$$

After getting the differential and substitute it in Equation (7), we have:

$$\begin{cases} \frac{\partial f(t)}{\partial t} = -\alpha f(t) - Q \\ f(T) = 0 \end{cases}$$
$$Q = e^{rt_0}$$
$$\alpha = \frac{1-\gamma}{2}((r + (\mu - r)^2/\sigma^2))$$
$$f(t) = \frac{Q}{\alpha}\left[e^{\alpha(T-t)} - 1\right] \quad (9)$$

The value of the function f is shown as a function of time. Substitute the risky assets; the following equation can be obtained.

$$\pi_1^* = \frac{\mu - r}{\gamma\sigma^2} x \quad (10)$$

Also, we obtained the optimal Lagrangian multiplier in Equation (11).

$$\lambda_1^* = (r - \mu)J_x - \sigma^2 J_{xx} \frac{k}{\sigma\varphi^{-1} + \mu dt} \quad (11)$$

3.2 Simulation and Numerical Solution

In this section, we consider the expected power utility function $\frac{{X_t}^{1-\gamma}}{1-\lambda}$ for wealth $0 < \gamma < 1$ and the expected return for risky assets by an investor $\mu > r$.

Furthermore, using numerical simulation in R Software and considering Equations (6) and (7), we obtain the optimal risky asset size. Besides, we will consider unconstrained solutions to equations (8), (9), and (10) as the initial value problem. To obtain the optimum value, investor's wealth and time divide into the computational domain $N_x \times N_t$ mesh points. The algorithm can be summarized as follows:

1) $\lambda^{*(0)} = 0$, $\pi^{*(0)}$, $J^{*(0)}$ are from the unconstrained solutions, and using Equations (8), (10), and (11), they are obtained.
2) For $x = [0, \Delta x, \dots, N_x]$, $t = [(N_t - 1)\Delta t, \dots, \Delta t, 0]$ we calculate the equation (12) for the amount $\lambda^{*(k+1)}$ and $\pi^{*(k+1)}$.

$$\pi_1^{*(k+1)} J_{xx}^{*(k)} \sigma^2 - J_x^{*(k)}(r - \mu) + \lambda_1^{*(k+1)} \left\{\frac{k}{\sigma\Phi^{-1}(\gamma_1) + \mu dt} - \pi_1^{*(k+1)}\right\} = 0$$
$$\lambda_1^{*(k+1)} \left\{\frac{k}{\sigma\Phi^{-1}(\gamma_1) + \mu dt} - \pi_1^{*(k+1)}\right\} = 0$$
$$\lambda_1^{*(k+1)} \leq 0 \quad (12)$$

For $j = [0,1, \dots, N_x]$, $n = [(N_t - 1), \dots, 1, 0]$ using derivation techniques for finite-state, Equations (13) are solved.

$$\frac{J_{j,n}^{*(k+1)} - J_{j,n}^{*(k+1)}}{\Delta t} + [(\mu - r)\pi_1^{*(k+1)} + rj\Delta x]\frac{J_{t,n}^{*(k+1)} - J_{j-1,n}^{*(k+1)}}{\Delta t} +$$

$$\frac{1}{2}(\pi_1^{*(k+1)})^2\sigma^2\frac{J_{j+1,n}^{*(k+1)} - 2J_{j,n}^{*(k+1)} + J_{j-1,n}^{*(k+1)}}{\Delta t} + (\frac{j\Delta x^{1-\gamma}}{1-\gamma}e^{-r\Delta t}) = 0 \quad (13)$$

4 Return to Step 2 until Convergence Problem

Remark that the optimization problem is in continuous time. For simplicity, we study at a fixed moment of time $t_0 = 0.5$, the motion of risky and non-risky assets processes as a function. Therefore, the above optimization problem is obtained as a sample at a specific moment in time, for the restricted and unrestricted risky assets. This strategy (to be static) in the previous section was considered. The value of the Lagrange multiplier as shadow prices for the imposed constraint in two cases are compared. The results obtained from the initial values $x = 2, T = 1, t_0 = 0.5, \mu = 0.15, r = 0.1, k = 0.05, \sigma = 0.2,$ $, \Phi^{-1}(\gamma_1) = 0.9, \Phi^{-1}(\gamma_2) = 0.3, \gamma = 0.5$ show in Figures 1 and 2. We assume that $N_t = 1000$ and $t = \frac{1}{1000}$.

We calculate the values of functions *f* and *g* with the substitute of the initial values of parameters with the optimal strategy for the portfolio. The value of risky assets is a linear function of total wealth at the time $t_0 = 0.5$, is $\pi_1^* = 2.5x$. This optimal strategy for the risky asset is shown in Figure 1. In this figure, the results of comparing constraint optimization problem (continuous line) and unconstraint optimization problem (stepwise line) are depicted. The horizontal axis shows the value of total wealth and the vertical axis shows the risky asset.

As this chart shows, when the limited price is imposed on a risky asset, the value of the risky asset changes to constant $\pi_1^* = 0.2775$ on the path of wealth. Also, in this case, we suppose the expected utility of the investor is a power utility function. Then imposing the constraint caused the value of the risky asset to increase to 0.27.

Similarly, by substituting the initial values of the Lagrange multiplier in Equation (11) $\lambda_1^* = -0.53x^{-0.5} + 0.003x^{-1.5}$ can be obtained. The results obtained for the Lagrange multipliers for the total amount of wealth, that is $x \geq 0.27$, is shown in Figure 2. Since the multiplier represents the shadow price of constraint on risky assets, it is in the negative zone before constraint is binding. This figure also confirms that the Lagrange multiplier is closer to zero when the constraint is binding.

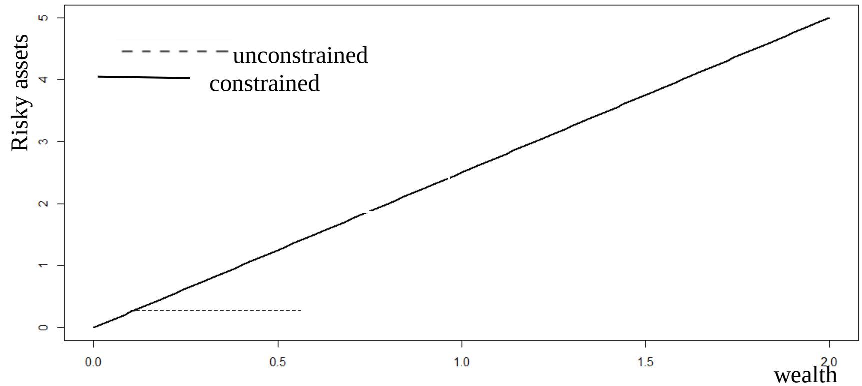

Figure 1. Comparing the Asset Value of Shares of Restricted and Unrestricted State

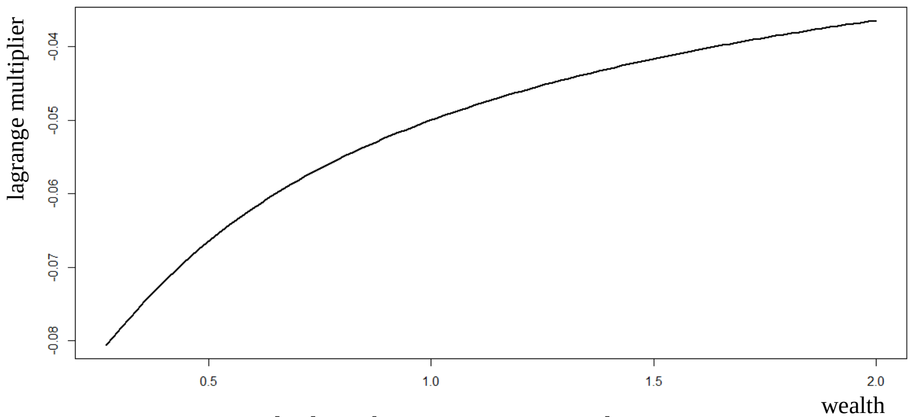

Figure 2. Lagrange Multiplier when Constraint is Binding.

5 Conclusion

Imposed price limits cause a new equilibrium path in stock price. This new equilibrium path will be different for investor's portfolios compared with no constraint in price movements. This study, by applying an expected power utility function assumption, confirms that imposed price limit would reduce investment in the risky asset and cause the equilibrium path different from the optimal path. We adopt simple assumptions on the parameters of the final HJB equation and used numerical simulation to obtain the equilibrium path at a specific time.

These results are at a certain time for the risk-free asset and risky asset with constant mean and variance. Also, if the parameters of the model change over time, the general framework for asset allocation is similar to the above approach, but to prove it, is a more complicated task.

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